BOWLAND MATHS
Professional development

The Case Studies and Mathematics
'Where is the maths in these Case Studies?'

Module overview

In the Case Studies, as in life, the situations are unstructured and the problems that arise have many alternative solutions. Pupils need to learn to *represent* and then *analyse* such situations using mathematics, *interpret and evaluate* the results, and *communicate* and *reflect* on their findings. This module is designed to help you consider how you can integrate and develop these Key Processes into your teaching.

This guide is intended for use alongside the *Bowland Maths DVD* or website, which include a short introductory video for each of the activities; longer videos of lessons and teacher discussions and links to all the handouts and ICT-based problems.

Introductory session	1 hour

- Look at a situation: Where is the maths?
- Look at the KS3 Key Processes
- Discuss some pedagogical implications
- Observe a lesson
- Plan a lesson using one of the problems.

Into the classroom	1 hour

- Introduce the situation then ask pupils to identify problems
- Simplify and represent the problem
- Review the representations pupils use
- Analyse and solve the problems
- Pupils communicate and reflect on their different approaches
- Review the Key Processes that pupils have been through

Follow-up session	1 hour
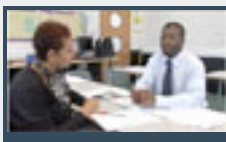
- Reflect on the lessons, and the ways maths emerged
- When should we introduce mathematical techniques?
- Integrating Case Studies into a scheme of work
- What about the tests?

Resources Needed

	Handout 1	Building a school with bottles in Honduras
	Presentation	BottleSchool (PowerPoint - optional)
	Handout 2	The modelling cycle
	Handout 3	The modelling cycle: questions to ask yourself
	Handout 4	Building a school with bottles: the Key Processes
	Handout 5	When should we introduce mathematical techniques?
	Handout 6	Typical maths activities in the Case Studies
	Handout 7	Types of problem used in the Case Studies
	Handout 8	What about the tests?
	Handout 9	Photographs for mathematical discussion
	Handout 10	Some mathematical questions on the photographs
	Handout 11	Suggested further reading

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Introductory session

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Activity 1	**Look at a situation: Where is the maths?**	**15 minutes**

It is not always easy for pupils to see any connection between the real world and mathematics lessons. As a result, they don't use the mathematics they learned in secondary school, even though thinking with mathematics could help them understand the world better – and make better decisions. The module begins with a real life context and looks at the mathematics that can arise from it.

Look at the photographs called *Building a school with bottles in Honduras* on **Handout 1**. This is presented as a context - no problems are posed.

Make a list of things you notice about the situation.
What mathematical questions occur to you?

You might begin by asking questions that start:
- How many ...?
- What would happen if ....?

Now set yourself a problem and use mathematics to tackle it.

Activity 2	**Look at the KS3 Key Processes**	**15 minutes**

Handout 2 - *The modelling cycle* shows the steps involved in modelling a real life situation. This flowchart shows the Key Processes used in the KS3 Programme of Study for mathematics (See: http://curriculum.qca.org.uk/subjects/mathematics/keystage3/).

Try to relate the work you have just done to *the modelling cycle* flowchart on **Handout 2.** How well does it fit?

Simplify and represent the situation:
- What specific problems did you pose?
- What simplifications and representations did you create?
- What choices did you make of information, methods and tools?

Analyse and solve the model you've made:
- What variables did you use?
- What information did you collect, or estimate?
- What relations between them did you formulate?
  What did you need to calculate, and how?

Interpret and evaluate the results:
- What did you learn about the situation? Were the results plausible?

Communicate and reflect on your findings:
- How could you best explain your analysis to someone else?
- What connections can you see to other problems?

Activity 3	Discuss some pedagogical implications	5 minutes

When pupils ask "Why are we doing this in maths?" they are reflecting a limited view of what mathematics is all about. Traditional styles of teaching can reinforce the impression that mathematics consists of little more than numbers and routine calculations. As the Key Concepts in the Programmes of Study for Mathematics shows, however, our aims are much broader than this. We are attempting to develop:

Competence in selecting and communicating appropriate mathematics.
Creativity when posing questions, constructing knowledge, tackling the unfamiliar.
Critical awareness and understanding of the applications, implications and limitations of mathematics.
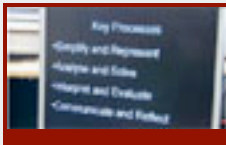
- How can you help *pupils* to become more aware of the Key Concepts in the Programmes of Study?
- How can you help *pupils* become more aware of the importance of the Key Processes shown in the modelling cycle on **Handout 2**?
- Should you explicitly discuss these goals with pupils?
- Should you gradually introduce pupils to the modelling cycle in pupil-friendly language?

Activity 4	Observe a lesson	10 minutes

Now watch Frank's lesson on *Building a School with plastic bottles.*
As you watch the lesson, ask yourself:

Which Key Processes can you see in the work of these pupils?

Can you see them:

Simplifying and representing the situation?
- What problems did they identify?
- What simplifications and representations did they create?
- What choices did they make of information, methods and tools?

Analysing and solving the model they've made?
- Which variables did they consider?
- What information did they collect, or guess?
- What relationships did they formulate?
- What calculations did they make?

Interpreting and evaluating the results?
- What did they learn about the situation?
- Were their results plausible?

Communicating and reflecting on the findings?
- How did they explain their analyses?
- What connections did they see to other problems?

Activity 5	Plan a lesson using one of the problems	15 minutes

Now it is your turn to plan a lesson using the *Building a School* situation.

Discuss how you will:

- introduce the situation to pupils;
- introduce the idea of the modelling cycle;
- organise the classroom and the resources needed;
- answer the question "Why are we doing this in maths?";
- conclude the lesson in a way that gives pupils a better understanding of the nature of mathematical processes.

It is helpful to present the lesson using a data projector. In addition, it is helpful to have a supply of the following resources available for working on the problems that arise:

- Some sample 1 litre plastic bottles
- Rulers or tape measures,
- Circular counters or coins (for working out how bottles pack together),
- Isometric dotted paper (to help with drawing and counting).
- Some copies of **Handout 3**: *The modelling cycle: questions to ask yourself* for pupils to use and discuss.

Photographs provide a powerful way of bringing real world contexts into the classroom. On **Handout 9** we have provided additional photographs that may be used to stimulate further mathematical work. You may prefer to collect your own, or use some from the resources suggested in the further reading at the end of this module. **Handout 10** contains some possible mathematical questions based on these photographs. What additional questions can you come up with?

This is the end of the *Introductory session*. After you have tried out your lesson with your own pupils, return for the *Follow-up session*.

Resources to support the lesson, and a suggested lesson plan, can be found in the *Into the classroom* session.

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The following suggestions describe one possible approach to using the photographs with pupils. This approach is intended to introduce them to the modelling cycle outlined in 📄 **Handout 3**. The timings below are very tentative. This lesson outline may well stretch into two lessons in practice!

Introduce the situation, then ask pupils to identify problems	5 minutes

> *The aim of today's lesson is to see if you can use mathematics to analyse a situation.*
> *To start with, you may not think the situation has anything at all to do with maths.*
> *I want to see if you can be creative and find ways of using the maths in your 'toolkit'.*

Introduce the situations carefully and vividly. Use the PowerPoint presentation on an interactive whiteboard, if possible.

> *These photographs were taken in Honduras. They show some people building a school out of old one-litre plastic bottles, just like the ones you buy lemonade in.*
> *They first fill them up with sand and then use them as bricks.*
> *This is a great way of using waste materials!*
> *What questions could we ask about this situation?*

Give pupils two minutes to note down any problems that spring to mind, then collect their ideas on the board. For example:

- *How many bottles (or how much sand) will it take to build one wall?*
- *How many bottles to build the whole building?*
- *How do the corners work?*

Ask pupils to identify which problems may be solved using mathematics and ask each group to choose one of these problems to work on.

Simplify and represent the problem	10 minutes

Explain that situations are sometimes too complicated to analyse as they stand. We have to simplify them before representing them with maths. Thinking with mathematics almost always involves this process.

> *How might we get started on the problem? Can we try a simpler problem first?*
> *What resources could we use to help us think about the problem?*
> *Would squared paper, isometric paper, a tape measure, a ruler help?*
> *What kinds of diagrams might help?*

Describe the resources that are available for working on the problem. Where appropriate, leave these at the side of the room, so that pupils can choose whether or not they use them.

Allow pupils 10 minutes to plan how they will work on the problems.

Right, now I'm giving you ten minutes to work on the problem in pairs. Then I'm going to ask some of you to come out and talk about the different approaches you are using.

Review the representations pupils use   10 minutes

Ask pupils to describe the methods and notations they are using. For example:

"We are simplifying the problem by looking at smaller walls and seeing if we can find a way of counting how many bottles will be needed. We are showing the bottles as black blobs.

This diagram shows that when there are 5 rows of bottles and the longest row contains 3 bottles, then 13 bottles are needed."
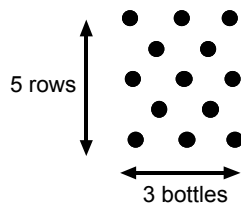
Of course, pupils may use all kinds of simplifications and notations and some may be more helpful than others. Spend some time discussing their advantages and disadvantages insofar as they are clear at this stage.

Analyse and solve the problems   20 minutes

Allow pupils time to work on the problems in pairs. As they do this, go round and offer general strategic guidance such as:

Take you time, don't rush.
What do you know?
What are you trying to find out?
Don't ask for help too quickly - try to think it out between yourselves.

To those who are struggling, ask appropriate questions from Handout 3:

Where have you seen something like this before?
Drawing this diagram out each time is taking you too long. Can you use a simpler representation?
What are you keeping fixed? What are you changing? Can you do this in a systematic way?
Can you see any patterns or relationships here? Can you explain them?
How can you keep a record of what you are doing?
Can you explain to me how this step follows this step?

For those who have made progress, move them towards interpretation and evaluation:

> *What have you found out so far?*
> *Convince me that your solution is a good one.*
> *How accurate is your answer? Is it accurate enough?*
> *Can you find another way that would give other ways of looking at the problem?*

Pupils communicate and reflect on their different approaches.   10 minutes

When most pupils have made significant progress with the problem, invite a few pairs of pupils to come to the front and communicate their ideas to the rest of the class. It does not matter if some have not yet reached any conclusions. They can still share their approaches and ideas.

> *Let's stop and share some of the different approaches we have used and consider what maths has been helpful and what unhelpful in each approach. Not everyone has finished, so I don't want to know about your answers; I want to hear your reasoning.*
>
> *Tell us about:*
> * *the problem you are solving;*
> * *how you have represented the problem as a mathematical model;*
> * *how you are analysing your model to get answers;*
> * *any conclusions you have reached so far. Do your answers make sense?*
>
> *We decided to find out how many bottles you would need for a building. We counted the bottles in one row, then the number of rows – but that wasn't easy to see. Then we multiplied those numbers. Then we said there were 4 walls, hopefully the same size. Then we began to worry about doors and windows...*

As pupils present their ideas, ask other pupils to comment on the advantages and disadvantages of each approach. If an explanation seems sound but is garbled, try:

> *Can you say that again please?*
> *You seem to have a good idea there but I want you to explain it*
> *as clearly as you can.*
> *Clear communication is important in mathematics.*

Review the Key Processes that pupils have been through   5 minutes

Introduce pupils to a simplified version of the modelling cycle and discuss the process they have been through. Try to make them a little more aware of the value of mathematical modelling.

> *Using mathematics involves all these processes. It is not just about learning simple techniques like how to add fractions! It is also about looking at situations in the world, simplifying them and analysing them to understand them better.*
>
> *This is what professional mathematicians do in their work.*

The Case Studies and Mathematics
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| Activity 1 | Reflect on the lessons, and the ways maths emerged | (15 minutes) |

Take some time to reflect on your own lesson and the Key Processes that were in evidence.

- What mathematical questions were identified?
- Did pupils use a range of mathematical representations?
- What relationships did they find in the situation?
- What calculations did they do? Could they interpret the meaning of these?
- Were they able to communicate their conclusions effectively?
- Did your pupils feel that this was different from a normal maths lesson?
- Are they now beginning to appreciate how the maths techniques they have studied may be linked to unfamiliar situations?

You will probably find that most of the tools that pupils chose to use was mathematics that they had been taught a year or two earlier. This is normal, for at least two reasons:

- The difficulty of a task reflects the *total cognitive load*, which depends on its *complexity*, its *unfamiliarity*, and its *technical demands.* If a task is complex and unfamiliar, pupils will use simpler techniques where they can.

- To be useful, concepts and skills have to be thoroughly absorbed and assimilated into the pupils' mathematical toolkit. This does not happen immediately but over time, through practice and, crucially, the building of multiple connections between topics and contexts of application.

| Activity 2 | When should we introduce mathematical techniques? | 15 minutes |

So far, we have been considering the role of the case studies in promoting Key Processes. The problems in the case studies also offer opportunities for developing mathematical *content knowledge*.

The Building a school situation offers excellent opportunities for pupils to develop competence in, for example, estimation, measurement, and calculations of area and perimeter. Pupils might make more progress if these topics are revised immediately *before* the situation is introduced. This might, however, constrain their thinking and reduce the task to an exercise in using these topics. On the other hand, these topics could be revised *during* or *after* working on the task, using the task to motivate the learning of technique.

- What are the advantages and disadvantages of each approach? Compare your thoughts with those given on
  📄 Handout 5: *When should we introduce mathematical techniques?*
- Which approach would you choose and why?

If the goal is to enable pupils to *choose* which skills to apply, then we must sometimes allow them the possibility of choosing suboptimal, inefficient methods and living with the consequences. If we always tell them which techniques to apply, then they will not develop autonomy in problem solving.

Activity 3	Integrating case studies into a scheme of work	20 minutes

The Bowland Case studies offer a range of activities which we hope will motivate and challenge your pupils. Although it is tempting to see them as activities for a "wet Friday afternoon" or for after the exams, this would not make the best use of their potential to enhance your scheme of work throughout Key Stage 3. To help you decide how they could be integrated into your teaching, this package includes a short portrait of each Case Study together with an analysis of the mathematics that it involves – the Key concepts, Key processes and the Mathematical topics.

Use 📄 **Handout 6** *Typical maths activities in the Case Studies* and 📄 **Handout 7** *Types of problem used in the Case Studies* to choose a Case Study you might like to use.

- Will the context of this case study interest and motivate the class?
- Will it offer variety of learning activity?
- Does it feature the Key Concepts and Key Processes that they need more of?
- Can it be tackled with mathematical concepts and skills they have been taught?
- Will it show and develop connections between these topics and with new contexts?
- Does it provide a starting point for further topic teaching in our scheme of work?

Activity 4	What about the tests?	10 minutes

All teachers work to help their pupils succeed in the national tests. This can deter teachers from spending time on open activities such as the Case Studies which appear very different from test questions.

In what ways do you think the Case Studies will help prepare pupils for the National Tests?

Discuss the points raised in 📄 **Handout 8** *What about the tests?*

Further Reading

See 📄 **Handout 11** for suggested further reading.

1   Building a school with bottles in Honduras

Look at the pictures and:
*   Make a list of things you notice.
*   Write down some mathematical problems that occur to you.
*   Now try to solve one problem!

First we collect old plastic bottles .....   and fill them with sand.

and make some foundations with rocks....   and start to build....

and build....   and build....

Add windows...   and plaster the walls.

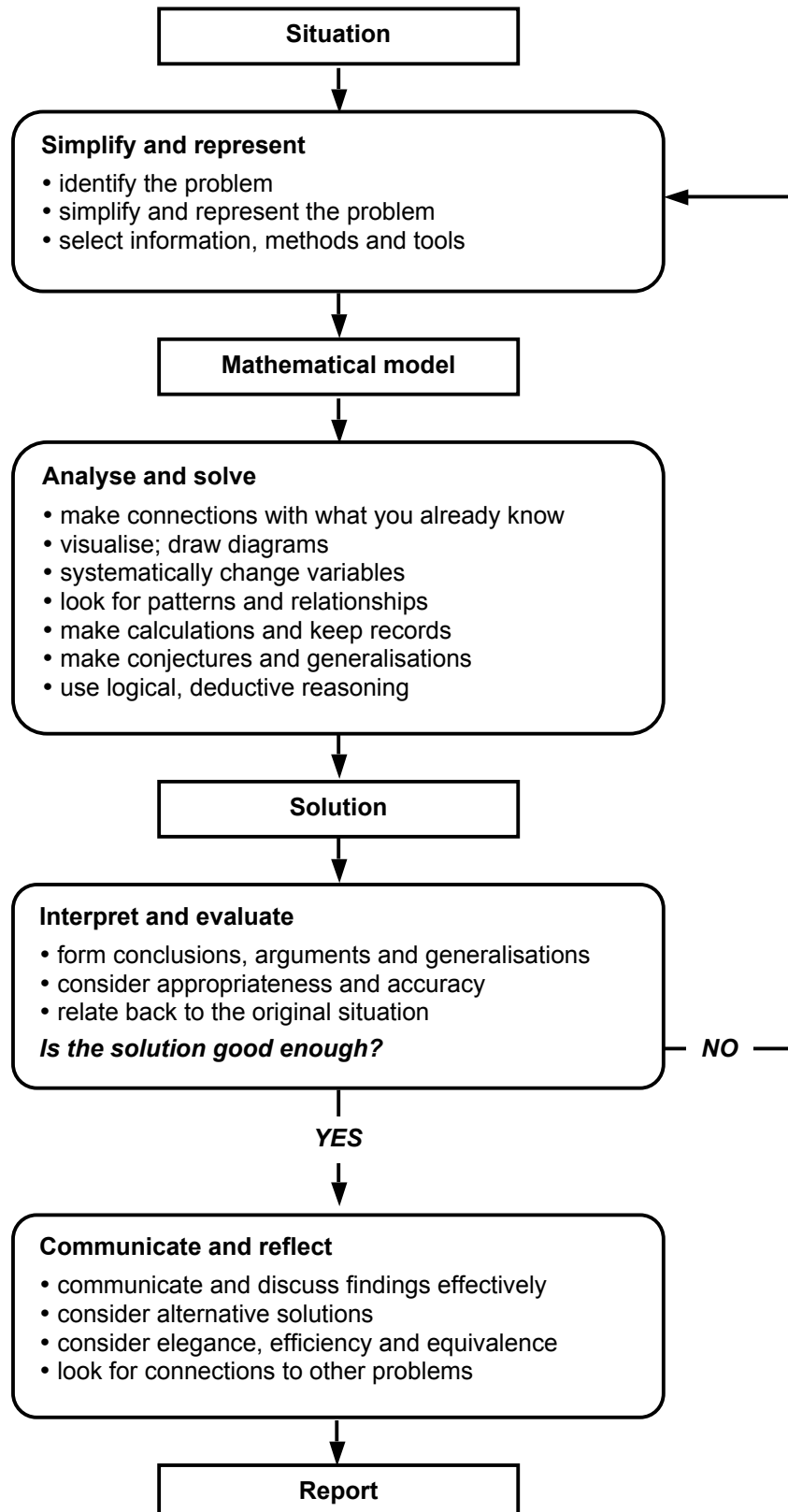This building is in Honduras and is now a centre for a secondary education programme that is designed to equip and motivate young people to help their communities and to reduce poverty. The programme is particularly designed to help students develop a capacity for problem solving.

Photographs with kind permission from:
Bayán Asociación de Desarollo Socio-Económico Indígena, La Ceiba, Honduras.

2   The modelling cycle

The narrow boxes represent states of the modelling process. The wide boxes describe the actions that move from one state to the next. These match the Key Processes in the Programmes of Study.

```
┌─────────────────────────────┐
│          Situation          │
└─────────────────────────────┘
                │
                ▼
┌─────────────────────────────────────┐
│  Simplify and represent             │
│  • identify the problem             │
│  • simplify and represent the problem │
│  • select information, methods and tools │
└─────────────────────────────────────┘
                │
                ▼
┌─────────────────────────────┐
│     Mathematical model      │
└─────────────────────────────┘
                │
                ▼
┌─────────────────────────────────────┐
│  Analyse and solve                  │
│  • make connections with what you already know │
│  • visualise; draw diagrams         │
│  • systematically change variables  │
│  • look for patterns and relationships │
│  • make calculations and keep records │
│  • make conjectures and generalisations │
│  • use logical, deductive reasoning │
└─────────────────────────────────────┘
                │
                ▼
┌─────────────────────────────┐
│          Solution           │
└─────────────────────────────┘
                │
                ▼
┌─────────────────────────────────────┐
│  Interpret and evaluate             │
│  • form conclusions, arguments and generalisations │
│  • consider appropriateness and accuracy │
│  • relate back to the original situation │
│  Is the solution good enough?       │── NO ──┐
└─────────────────────────────────────┘        │
                │ YES                           │ (back to Simplify and represent)
                ▼
┌─────────────────────────────────────┐
│  Communicate and reflect            │
│  • communicate and discuss findings effectively │
│  • consider alternative solutions   │
│  • consider elegance, efficiency and equivalence │
│  • look for connections to other problems │
└─────────────────────────────────────┘
                │
                ▼
┌─────────────────────────────┐
│           Report            │
└─────────────────────────────┘
```

3   The modelling cycle: questions to ask yourself

Situation

Simplify and represent

- What exactly is the problem? Can I say it clearly?
- What do I know, and what do I need to find out?
- Can I make the problem a bit simpler?
- What assumptions can I make?
- What bit of maths might help me?

Mathematical model

Analyse and solve

- Have I seen something like this before?
- Is there a diagram that I could draw?
- What is fixed, and what can I change?
- Can I do this in an organised way?
- How can I keep a record of what I am doing?
- Can I see any patterns or relationships?
- Will this pattern always work? How can I be sure?

A solution to the problem

Interpret and evaluate

- What conclusions can I draw?
- How can I check my work?
- Are my conclusions reasonable?

Is my solution good enough?

NO

YES

Communicate and reflect

- What is the best way to show my work to others?
- Looking back, was there a better way to do it?
- Will this work be useful for other problems?

Report

4   Building a school with bottles: the Key Processes

Below we illustrate some of the mathematical potential of the situation, referring to the Key Processes in the KS3 Programme of Study.

(i) Simplify and Represent

We first identify some of problems that may be asked:
- How many bottles do I need for a building like this?
- How tall is the building, and the man?
- How do the bottles fit together?
- How much sand will we need to fill the bottles?
- What about the mortar in between?
- How do the corners work?
- What about doors and windows?
- What about the roof?

We'll focus (to begin with, at least) on a practical approach to
How many bottles do I need for a building like this?

To begin with we'll simplify the situation to assume there are 4 walls (as suggested by the angles in the bottom photograph), all the same size, and that there are no windows! We'll make calculations easier if we also assume that the number of bottles needed would not be much different if they were stacked in a 'square' fashion: i.e.
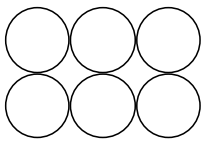
like this...   rather than like this...
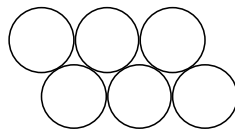
We'll modify these assumptions in the second cycle of the process.

(ii) Analyse and solve

Count the number of bottles in a row.
Estimate the number of rows (you can't see them all)
Number in one wall is approximately the product of these.
Add up for 4 walls – assume the walls are the same size.

There are about 25 bottles in a row.
We can see and count only the top 7 rows clearly; these are about 1/3rd of the height
So we estimate that there are about 3 x 7 ~ 20 rows
So the wall contains about 25 x 20 ~ 500 bottles
Assuming the 4 walls are the same size gives 4 x 500 = 2000 bottles

(iii) Interpret and evaluate

This is good enough to illustrate the modelling process (and easy to report), but (and this is why it is a modelling *cycle*) if we were really serious about understanding the problem it would need to be improved by returning to tackle some of the other questions listed above.

Possible refinements include, for example:
- What size bottles are these? (We could estimate from the height of the man)
- How much sand would we need?
  (E.g., 2000 one-litre bottles need 2-3 tonnes; why?)
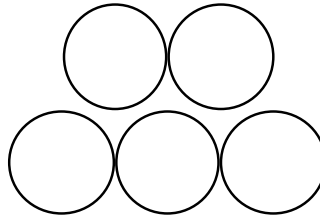- …. and, of course, we would need to make a proper plan for the building

(i) Simplify and represent

We could represent the stacking of the bottles in other ways, for example by closer packing like figure A (assuming no mortar) or figure B (with some mortar).
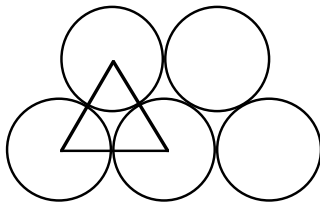
No mortar: Some mortar between rows:

(ii) Analyse and solve

If there was no mortar, the length of the longest row would be equal to the *diameter of bottle* x *number of bottles in a row*. The height between rows would be the height of the equilateral triangle in the figure. This might be calculated either by Pythagoras or simply by measuring a model made from three bottles!
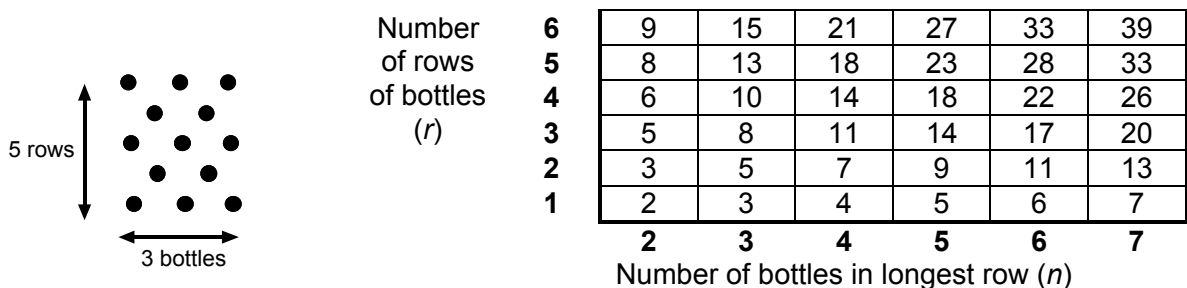
Height between rows $= \dfrac{\sqrt{3}}{2} \times diameter \approx 0.87 \times diameter$

So the saving in gaps from close packing (compared with square stacking) would be about 13% although there are bigger gaps at the ends of each row.

With mortar, the height between rows appears to be approximately equal to the diameter of each bottle. Thus we can reasonable assume that the height of a wall is approximately equal to the *diameter of bottle* x *number of rows of bottles.*

Both models reduce the number of bottles needed by only one for every two rows.

The number of bottles needed for each wall may be counted and represented in a table:

Number of rows of bottles (r)		2	3	4	5	6	7
	6	9	15	21	27	33	39
	5	8	13	18	23	28	33
	4	6	10	14	18	22	26
	3	5	8	11	14	17	20
	2	3	5	7	9	11	13
	1	2	3	4	5	6	7

Number of bottles in longest row (n)

If we assume (as before) that there are 25 bottles in the longest row and 20 rows, then this arrangement would require just 10 fewer bottles, or 490 bottles for each wall.
For 4 walls this gives 1960 bottles - only 2% fewer than our previous estimate!

(iii) Interpret and evaluate

This analysis confirms our earlier one as being very reasonable.
The following analysis is algebraic, an approach that reveals the general structure of the problem. This will be beyond the capabilities of many pupils, but it illustrates here the process of analytic modelling in a simple situation.

(i) Simplify and represent

How many bottles do I need to build any rectangular building of bottles?

First **select and list the variables**:

Height of wall h
Width of wall w
Diameter of a bottle d
Number in row n
Number of rows r
Number in wall W
Total number of bottles T
We will also denote each wall by subscripts 1 to 4.

Now we **generate relationships** between the variables:

$T = W_1 + W_2 + W_3 + W_4$ (Total number of bottles in all 4 walls)

$W_1 = n \times r$ etc.   (Assume all walls have same height, thus same r)

$r = \dfrac{h}{d}$ (Assume rows are d apart)

$n_1 = \dfrac{w_1}{d}$ etc.   (Not assuming each wall is same width)

(ii) Analyse and solve

We can form some new equations by combining these:

$W_1 = n \times r = \dfrac{w_1}{d} \times \dfrac{h}{d} = \dfrac{w_1 \times h}{d^2}$

$T = (w_1 + w_2 + w_3 + w_4) \times \dfrac{h}{d^2}$

$T = P \times \dfrac{h}{d^2}$ (where P = Total perimeter of house)

$T = \dfrac{A}{d^2}$ (where A = Total area of the walls)

(iii) Interpret and evaluate

We can get estimates for the number of bottles needed from either of these two last equations. The final one also doesn't assume that there are no doors and windows.
It simply states that each bottle occupies a wall area equal to the square of its diameter.
Perhaps we should have seen this simple relationship at the outset!

5   When should we introduce mathematical techniques?

Some teachers are discussing a case study that will take 3-5 maths lessons.
They decide that pupils will make more progress if they have a sound knowledge of *X*,
where *X* represents any technique or area of knowledge.

The teachers are trying to decide whether to teach *X* before, during or after working on the
case study:

Before? "I'll teach them about X in the week before we do the case study, so that when we come to do the case study, pupils will be able to apply this technique/knowledge."	**Advantage**: pupils will have techniques polished and ready to use. **Danger:** case study becomes an exercise in technique, rather than an opportunity to develop autonomous problem solving strategies.
During? "We'll start the case study, and if pupils get stuck, we'll break off working on the case study for a lesson or two, and I'll give them practice with X.	**Advantage**: You can respond to needs as they arise. **Danger:** if pupils expect you to bale them out when the going gets difficult, you reinforce dependence and undermine autonomy
After? "We'll attempt the whole case study and I'll see how pupils get on. Afterwards, I will introduce them to X and refer back to the case study to show them what a powerful idea it is."	**Advantage**: The experience of working on a case study may motivate and enable pupils to perceive the value of techniques when they are taught. **Danger:** Pupils may still not be able to use techniques autonomously, unless they are given further opportunities to apply them in further case studies.

6   Typical Mathematics activities in the Case Studies

The table shows the case studies available at the time of writing, with examples of the mathematical activities involved. More detailed information is given in *Portraits of the Case Studies.*

Alien invasion	Locating spaceships using clues involving distances and directions. Cracking a code to escape from a cell.
Crash test	Controlling variables systematically (e.g. speeds, design of cars, barrier types). Making hypotheses and testing them by observing the effects in crash test experiments. Presenting findings to the class.
Explorers	Planning a route bearing in mind fuel, food reserves and distance. Trading between planets using fantasy units of currency. Using algebraic functions to decide where charges should be placed to destroy asteroids.
Highway link design	Proposing the location of a by-pass, using data tables and graphs used by the Highways Agency. Satisfying constraints (minimum radii of curvature, verge clearance, cambers etc). Costing and presenting proposed solutions.
How risky is life?	Comparing people's perceptions of the causes of death with the actual statistics. Interpreting very large and very small probabilities. Deciding what these say about our behaviour and attitudes. Exploring random variation.
Keeping the pizza hot	Choosing packaging for a pizza. Measuring temperatures as the pizza cools. Using data logging software. Fitting a graphical model to the cooling of a pizza. Calculating longest reasonable travel time before a pizza becomes too cold to eat.
My music	Describing the characteristics of individual genres of music. Using the tempo of music to illustrate the creation of a compound measure, beats per minute.
Mystery tours	Planning a 5-day trip to satisfy constraints of money/time and keep all the tourists happy. Converting currencies, satisfying baggage allowances etc.
Outbreak	Using coordinate clues to locate infected people. Mixing ingredients in correct proportions to create an antidote. Using resources optimally to design a vaccination programme.
PointZero	Solving number, spatial and logic puzzles to progress in an adventure game. Using number sequences to escape from a building. Using rotation and reflection to recreate a given pattern. Using codes and loci to escape from underground tunnels.
Product wars	Designing a questionnaire and conducting market research, Mixing ingredients to obtain optimum nutritional value and taste; designing the packaging for the drink.
Reducing road accidents	Exploring one town's accident database. Controlling variables to decide how a given sum of money should be allocated on safety measures. Preparing a case and presenting it convincingly.
Save a baby kangaroo	Determining the age and species of a 'Joey' from tail and foot measurements and graphs of growth data. Devising an appropriate nutrition regime from tables of nutrient data. Presenting this regime.
Speed cameras	Exploring perceptions of randomness and relating this to the perceived effectiveness of speed cameras. Simulating the effects of different sitings
Water availability	Analysing a complex decision faced by a water aid agency; Devising and using a compound measure (eg per capita) to decide on a 'fair' distribution of resources.
You reckon?	Breaking a problem up into component parts; combining everyday knowledge to create chains of reasoning that result in reasonable estimates of useful quantities.

7   Types of problem used in the Case Studies

Type of problem	Typical examples found in the case studies
Planning and organising Find optimum solution subject to constraints.	**Outbreak** Mix ingredients to create an antidote., devise a vaccination programme. **Product wars** Mix ingredients to obtain optimum nutritional value and taste **Mystery tours** Plan a tour to satisfy time/money/customers **Highway link design** Propose the optimum location of a by-pass using data used by the Highways agency.
Designing and making Design an artifact or procedure and test it	**Product wars** Package a drink and test it through market research
Modelling and explaining Create notations and models and use them to explain phenomena or propose solutions	**Water availability** Create a fair way to distribute water. **My music** Create a measure of tempo. **Keeping the pizza hot** Model the cooling of a pizza
Exploring and discovering relationships Find relationships, estimate and predict results and test them.	**Crash test** Exploring the effect of different variables when crash testing cars. **Speed cameras** Investigating the effects of different sites for speed cameras. **How risky is life?** Estimate risks and test them against real data.
Interpreting and estimating Deduce information, from representations of data and present a reasoned argument.	**Save a baby kangaroo** Devising an appropriate nutrition regime from tables of nutrient data. Present this regime. **Reducing road accidents** Exploring one town's accident database; use graphs, tables and charts to construct a case. Present the case. **You reckon?** Make reasoned estimates to test common assertions and facts.
Solving logic puzzles Here the contexts are more 'fantasy' and embedded in computer games	**Alien invasion** **PointZero** **Explorers**

8   What about the tests?

Will working on the Case Studies help to improve pupil's scores in "high stakes" KS3 tests, even though the tasks in the tests are so different?

There are three main reasons for believing that this is true:

- **The tests are changing**
  The tests are now being redesigned to match the newly revised National Curriculum Programmes of Study for Key Stages 3 and 4. These case studies address all the aspects of these, including the Key Concepts and Key Processes which require pupils to sustain substantial chains of reasoning working from *representing* a problem with mathematics, *analysing* this mathematical model to find solutions, *interpreting and evaluating* these solutions in the problem context and *communicating* the results and the reasoning that produced them. This is a broader range of mathematical performance than current tests, which concentrate on short 'items' that assess separate concepts and skills. Work on the case studies will give teachers a head start on this broader range of performance, as well as equipping pupils better for their future lives.

- **Connections build long term learning**
  There is a deeper reason for using rich problems in the mathematics classroom. They will improve understanding of basic concepts and skills by helping pupils build *multiple connections,* within and between topics and practical contexts. It is these links that give strength and robustness to conceptual understanding, reducing the fading grasp that every teacher knows so well, and saving the consequent time used for re-teaching.

  Only single connections arise naturally in the normal linear process of teaching, where one topic is linked to the previous one. In exploring more open situations, pupils begin to see multiple connections, as they select tools from their mathematical toolkit that will help them tackle a problem they have not met before. There was clear evidence of this in the *Building a school with bottles* situation, where pupils were linking different topics: estimation, measurement, areas, perimeters and so on.

- **Substantial problems improve motivation**
  For a few, mathematics itself is fascinating enough, particularly if brilliantly taught. For most people, using mathematics to gain power over problems in practical contexts, from the real world and from fantasy domains, motivates them to learn more. Teachers of English have long exploited this opportunity, which has been neglected in mathematics teaching. The cases studies help fill this gap.

9   Photographs for mathematical discussion

Look at each of the photographs below and, for each one:

- Make a list of things you notice.
- Write down some mathematical problems that occur to you. They might, for example, start like this:
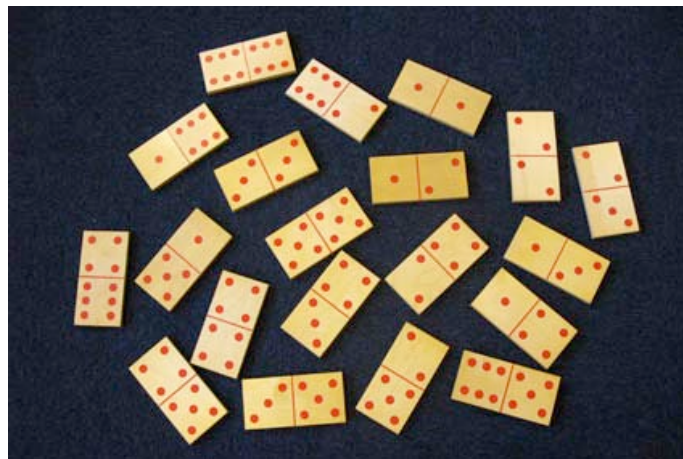
    How could I describe ..... ?
    How many ...?
    What would happen if I changed ....?

Now do some mathematics based on the photograph!

Dominoes

Calendar

Stack of barrels

A pavement in Germany

Trike with square wheels

Russian dolls

These photographs were taken by Malcolm Swan.

Further photographs leading to interesting mathematical discussions may be obtained
from Richard Phillips at http://www.problempictures.co.uk/

10   Some mathematical questions on the photographs

Dominoes

This appears to be part of a set that includes (1,1) to (6,6) - no blanks.
- Which domino is missing?
- How can you organise the dominoes systematically?
- Can you make a chain with the complete set? How can maths help?
- Can you make a ring with the complete set?
- How many spots are there altogether in a complete set?
  What is a quick way of counting them?
- How many dominoes are there in a complete set from (1,1) to (n,n)?

Calendar

- How are the numbers arranged on the cubes?
- Can you draw nets and make the cubes?
- What impossible dates can be made from these cubes?

Stack of barrels

- How many barrels are in the stack?
- If you make a taller stack 4, 5, ... barrels high, how many barrels will you need?
- Generalise?
- How else could you stack these barrels? What other pyramids are possible?

A pavement in Germany

- What shapes can you see?
- Are all the paving slabs identical? What shape are they?
- Can you work out any angles?
- Can you draw one of the slabs accurately?
- Can you find other pentagons that tessellate?
- What other shapes can paving slabs be?
  Make up some an interesting shape of your own and show how it can tile.

Trike with square wheels

- Does the trike run smoothly?
- Can you make a simple model?
- What is the height of each 'bump' on the track?
- Can you draw the shape of the 'bumpy road' accurately?
- What would happen if you had triangular wheels or hexagonal wheels?

Russian dolls

- Do the tops of the heads lie on a straight line?
  What does this tell you?
- If you divide each doll's height by its width, what do you get?
  What does this tell you?
- If you were to make some bigger dolls in this set - how big would they have to be?

11   Suggested further reading

Learning mathematics through contextualised situations.
Boaler J. (1993) 'The Role of Contexts in the Mathematics Classroom', *For the Learning of Mathematics* 13(2)

Looking at the apprenticeship model of learning.
Brown, J. S., Collins, A. and Duguid, P. (1989) 'Situated cognition and the Culture of Learning', Educational Researcher, 18 (1), pp 32-42.

Looking at a different way to organise the Year 9 curriculum
Carter, C. (2008) 'A different way', *Mathematics Teaching,* 207, pp 38-40
http://www.atm.org.uk/mt/archive/mt207files/ATM-MT207-38-40-mo.pdf

What do pupils see as mathematical? Does it have to have numbers?
Mendick, H., Moreau, M. and Epstein D. (2007) 'Looking for mathematics' in D. Kuchemann (Ed.) Proceedings of the British Society for Research into Learning Mathematics 27 (1) pp 60 – 65
http://www.bsrlm.org.uk/IPs/ip27-1/BSRLM-IP-27-1-11.pdf

A comparison of the mathematics people use in school and out of school.
Nunes, T., Schliemann, A.D., Carraher, D.W. (1993), *Street mathematics and school mathematics,* Cambridge University Press

What is important in mathematics education?
Polya G (2002) 'The goals of mathematical education: part 1 and part 2' *Mathematics Teaching,* 181, pp 6-7 and 42-44
http://www.atm.org.uk/mt/archive/mt181files/ATM-MT181-06-07.pdf
http://www.atm.org.uk/mt/archive/mt181files/ATM-MT181-42-44-mo.pdf